MT308 Numerical Analysis

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Taylor's Theorem

Taylor's theorem is fundamental to numerical analysis for many reasons:

- Approximating functions by polynomials.
- error estimation
- Theoretical basis for some methods (like Newton's method)
- Numerical differentiation
- Numerical integration
- Solving differential equations

Therefore, it is useful to recall Taylor's theorem, which you studied in your Calculus classes.

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Taylor's Theorem

Taylor's theorem expands a differentiable function into a polynomial plus a remainder term: $f(x) = P_k(x) + R_k(x)$

Let x and x_0 be real numbers, and let f be k + 1 times continuously differentiable on the interval between x and x_0. Then there exists a number c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

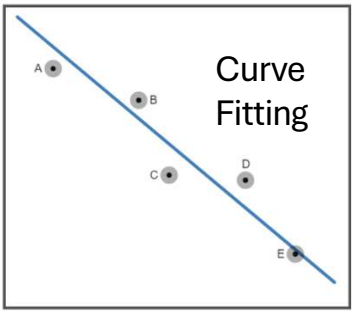
The polynomial part of the result ($P_k(x)$), the terms up to degree k in x − x_0, is called the degree k Taylor polynomial for f centred at x_0. The final term ($R_k(x)$) is called the Taylor remainder.

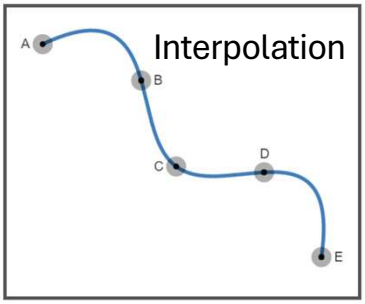
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Interpolation and curve fitting

- Given a set of data, we might want to approximate the value in between, after, or before what is given. To do this, we can generate a curve by using either **interpolation** or **curve fitting**.
- **Interpolation** is a method of estimating the value of a function at a given point within the range of a set of known data points. It involves finding a polynomial *that* ***fits a set of data points exactly***, rather than just approximating them.
- **Curve fitting**, on the other hand, is the process of finding the best-fitting curve, where the goal is to find a model that captures the *underlying* ***trends in the data***, rather than fitting the data points exactly.

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Curve Fitting is used for:

- Trend analysis. Predicting values of dependent variable, may include extrapolation beyond data points or interpolation between data points.
- Hypothesis testing. Comparing existing mathematical model with measured data.

Interpolation is used for:

- Estimate missing data points between known values.
- Create smooth curves or functions from discrete data sets.
- Resizing images in image processing
- Animating motion by calculating intermediate frames

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Interpolation

- Interpolation is a mathematical technique used to estimate unknown values within the range of a discrete set of known data points.
- Interpolation is the process of finding a value between two known values in a dataset.
- Finding an estimate $f(x)$ for some x within the range $[x_0,x_n]$ that passes through the data points (x_0,y_0), (x_1,y_1),...(x_n,y_n) .
- Interpolation helps approximate data or function values where direct evaluation is difficult or unavailable.

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Types of Interpolation Methods

- **Polynomial Interpolation**:
  - *Lagrange Interpolation*
  - *Newton's Divided Differences*
- **Piecewise Interpolation**:
  - *Linear Interpolation*
  - *Cubic Spline Interpolation*
- **Other Methods**:
  - *Hermite Interpolation*
  - *Barycentric Interpolation*

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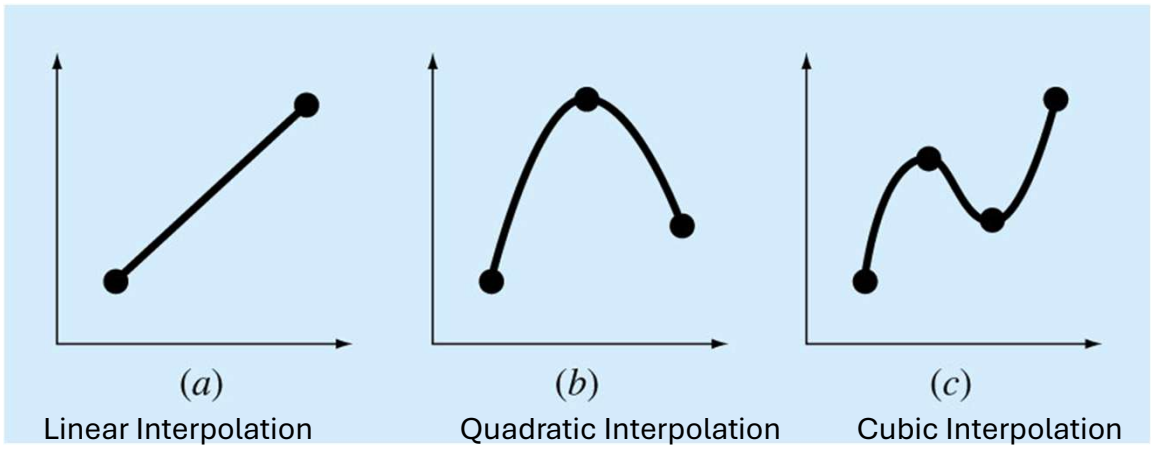
Polynomial Interpolation

- Estimation of intermediate values between precise data points. The most common method is to fit a polynomial function to the known point:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

- Although there is one and only one *n*th-order polynomial that fits *n+1* points, there are a variety of mathematical formats in which this polynomial can be expressed:
  - The Newton polynomial
  - The Lagrange polynomial

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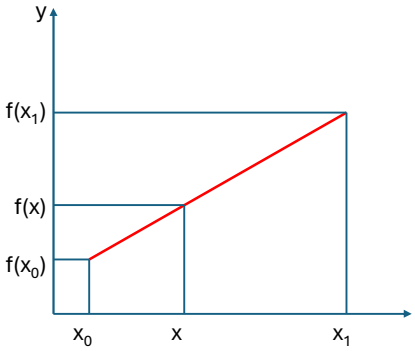


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Newton's Divided-Difference Interpolating Polynomials

Linear Interpolation is the simplest form of interpolation, connecting two data points with a straight line:

The Slope $= \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x)-f(x_0)}{x-x_0}$



Then $f(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$

Linear-interpolation formula

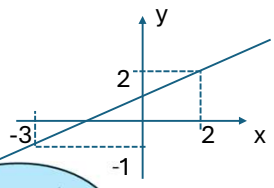
Slope and a finite divided difference approximation to 1st derivative

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Example: Find an interpolation polynomial for the data points (-3, 2) and (2, -1). Since there is only 2 points, this is the same as "Find the equation of the line that passes through the points (-3, 2) and (2, -1)".

$y = mx + c$

$m = \frac{2-(-1)}{2-(-3)} = \frac{3}{5}$



$c = \frac{1}{5}$ and $y = -\frac{3}{5}x + \frac{1}{5}$

Using Newton's interpolation formula

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= 2 + \frac{-1-2}{2-(-3)}\left(x-(-3)\right)$$

$$= 2 - \frac{3}{5}(x+3) = -\frac{3}{5}x + \frac{1}{5}$$

The same result is found in both ways: Finding the equation of the line by using the slope of the line and using Newton's formula for linear interpolation

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Quadratic Interpolation

- If three data points are available, the estimate is improved by introducing some curvature into the line connecting the points.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- A simple procedure can be used to determine the values of the coefficients.

$$x = x_0 \qquad b_0 = f(x_0)$$

$$x = x_1 \qquad b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x = x_2 \qquad b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

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Example: Find an interpolating polynomial for the data points (0,1),(2,2), and (3,4)

$x_0 = 0$ and $b_0 = f(0) = 1$

$x_1 = 2$ and $b_1 = \frac{f(2)-f(0)}{2-0} = \frac{2-1}{2} = \frac{1}{2}$

$x_3 = 3$ and $b_2 = \frac{\frac{f(3)-f(2)}{3-2} - \frac{f(2)-f(0)}{2-0}}{3-0} = \frac{(f(3)-f(2))-b_1}{3-0} = \frac{4-2-\frac{1}{2}}{3} = \frac{1}{2}$

Then $f(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2) = 1 - \frac{1}{2}x - \frac{1}{2}x^2$

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General Form: Newton's Interpolating Polynomials

$$f_n(x) = f(x_0) + (x-x_0)f[x_1,x_0] + (x-x_0)(x-x_1)f[x_2,x_1,x_0] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_n,x_{n-1},\cdots,x_0]$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_1,x_0]$$

$$b_2 = f[x_2,x_1,x_0]$$

$$\vdots$$

$$b_n = f[x_n,x_{n-1},\cdots,x_1,x_0]$$

$$f[x_i,x_j] = \frac{f(x_i)-f(x_j)}{x_i-x_j}$$

$$f[x_i,x_j,x_k] = \frac{f[x_i,x_j]-f[x_j,x_k]}{x_i-x_k}$$

$$\vdots$$

$$f[x_n,x_{n-1},\ldots,x_1,x_0] = \frac{f[x_n,x_{n-1},\ldots,x_1]-f[x_{n-1},x_{n-2},\ldots,x_0]}{x_n-x_0}$$

Bracketed function evaluations are finite divided differences

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Errors of Newton's Interpolating Polynomials

Structure of interpolating polynomials is similar to the Taylor series expansion in the sense that finite divided differences are added sequentially to capture the higher order derivatives.

- For an n^{th}-order interpolating polynomial, an analogous relationship for the error is:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

ξ Is somewhere containing the unknown data

- For non differentiable functions, if an additional point $f(x_{n+1})$ is available, an alternative formula can be used that does not require prior knowledge of the function:

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x-x_0)(x-x_1)\cdots(x-x_n)$$

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Lagrange Interpolating Polynomials

- The Lagrange interpolating polynomial is simply a reformulation of the Newton's polynomial that avoids the computation of divided differences:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

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$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

•As with Newton's method, the Lagrange version has an estimated error of:

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i)$$

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Example: Find an interpolating polynomial for the data points (0,1),(2,2), and (3,4)

Using the second-order Lagrange interpolating polynomials

$$\begin{aligned} f_2(x) &= 1\frac{(x-2)(x-3)}{(0-2)(0-3)} + 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)} \\ &= \frac{1}{6}(x^2 - 5x + 6) + 2\left(-\frac{1}{2}\right)(x^2 - 3x) + 4\left(\frac{1}{3}\right)(x^2 - 2x) \\ &= \frac{1}{2}x^2 - \frac{1}{2}x + 1. \end{aligned}$$

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Coefficients of an Interpolating Polynomial

- Although both the Newton and Lagrange polynomials are well suited for determining intermediate values between points, they do not provide a polynomial in conventional form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_x x^n$$

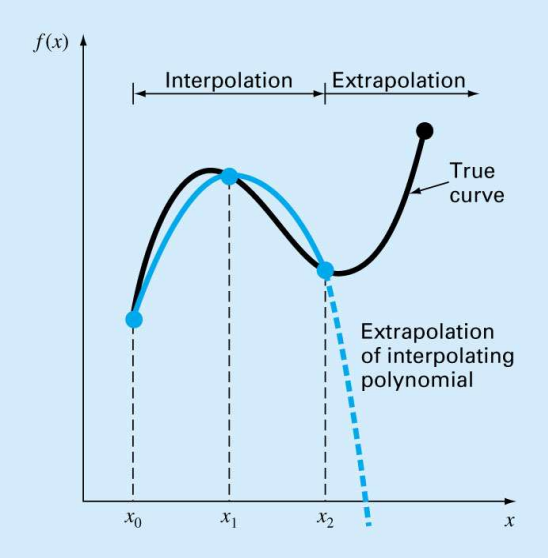
- Since $n+1$ data points are required to determine $n+1$ coefficients, simultaneous linear systems of equations can be used to calculate "a"s.

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$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 \cdots + a_n x_0^n$$
$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 \cdots + a_n x_1^n$$
$$\vdots$$
$$f(x_n) = a_0 + a_1 x_n + a_2 x_n^2 \cdots + a_n x_n^n$$

Where "x"s are the knowns and "a"s are the unknowns.

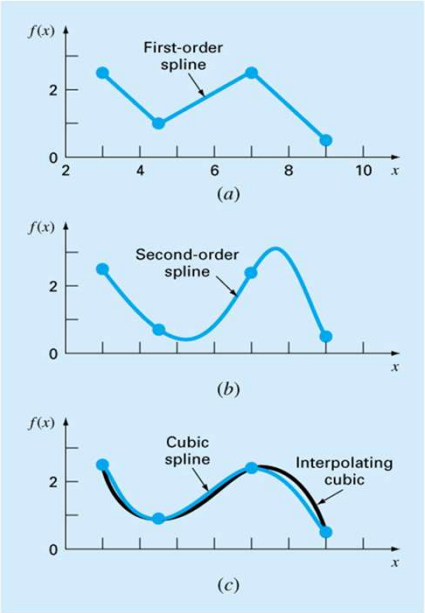
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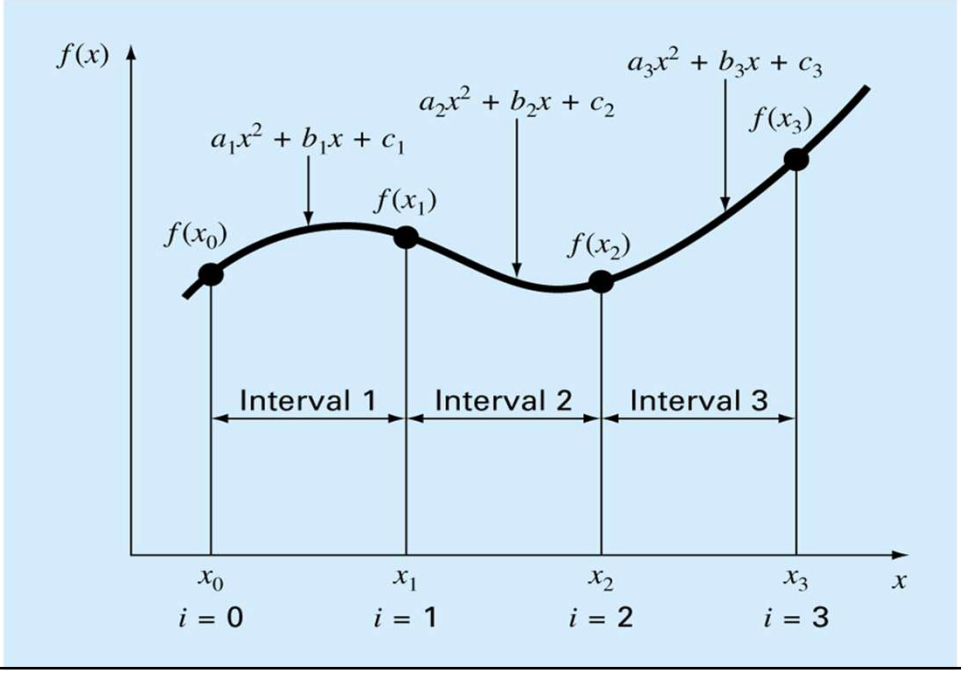
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Spline Interpolation

- There are cases where polynomials can lead to erroneous results because of round off error and overshoot.
- Alternative approach is to apply lower-order polynomials to subsets of data points. Such connecting polynomials are called spline functions.



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Example (Assignment 9) Find f(-1), f(1) and f(5) given that f(x) passes through (-2, 4.14), (0, -5.5), (2, 6.74), (4, 9.26) and (6, -3.18).

This is a typical interpolation problem. Without specific requirements, we may solve this in various ways, including the following methods:

Solution 1: Piecewise linear interpolation.

Use $f(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$

Consider (-2, 4.14) and (0, -5.5) then $f(x) = 4.14 + \frac{-5.5-4.14}{0-(-2)}(x-(-2))$ then $f(x) = -4.82x - 5.5$ and $\boldsymbol{f(-1) = 4.82 - 5.5 = -0.68}$

Consider (0, -5.5) and (2, 6.74) then $f(x) = -5.5 + \frac{6.74-(-5.5)}{2-0}(x-0)$ then $f(x) = 6.12x - 5.5$ and $\boldsymbol{f(1) = 6.12 - 5.5 = 0.62}$

Consider (4, 9.26) and (6, -3.18) then $f(x) = 9.26 + \frac{-3.18-9.26}{6-4}(x-4)$ then $f(x) = -6.22x$+34.14 and $\boldsymbol{f(5) = 6.12 - 5.5 = 3.04}$

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Example (Assignment 9) Find f(-1), f(1) and f(5) given that f(x) passes through (-2, 4.14), (0, -5.5), (2, 6.74), (4, 9.26) and (6, -3.18).

Solution 2: Use Lagrange interpolating polynomials to assign a 4th order polynomial to the given points: (*Note: longest but the best approximation*)

$$f(x) \cong P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) + L_4(x)f(x_4)$$

Since $L_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{4} \frac{(x-x_k)}{(x_i-x_k)}$ then

$$L_0(x) = \frac{(x-0)}{(-2-0)}\frac{(x-2)}{(-2-2)}\frac{(x-4)}{(-2-4)}\frac{(x-6)}{(-2-6)} = \frac{x^4-12x^3+44x^2-48}{384} = \frac{x^4}{384} - \frac{x^3}{32} + \frac{11}{96}^2 - \frac{x}{8}$$

$$L_1(x) = \frac{(x+2)}{(0+2)}\frac{(x-2)}{(0-2)}\frac{(x-4)}{(0-4)}\frac{(x-6)}{(0-6)} = \frac{x^4-10x^3+20x^2-40x-96}{-96} = \frac{-x^4}{96} + \frac{5x^3}{48} - \frac{5x^2}{24} + \frac{5x}{12} + 1$$

$$L_2(x) = \frac{(x+2)}{(2+2)}\frac{(x-0)}{(2-0)}\frac{(x-4)}{(2-4)}\frac{(x-6)}{(2-6)} = \frac{x^4-8x^3+4x^2-40x}{64}$$

$$L_3(x) = \frac{(x+2)}{(4+2)}\frac{(x-0)}{(4-0)}\frac{(x-2)}{(4-2)}\frac{(x-6)}{(4-6)} = \frac{x^4-6x^3-4x^2-40x}{-96}$$

$$L_4(x) = \frac{(x+2)}{(6+2)}\frac{(x-0)}{(6-0)}\frac{(x-2)}{(6-2)}\frac{(x-4)}{(6-4)} = \frac{x^4-4x^3+4x^2-40x}{384}$$

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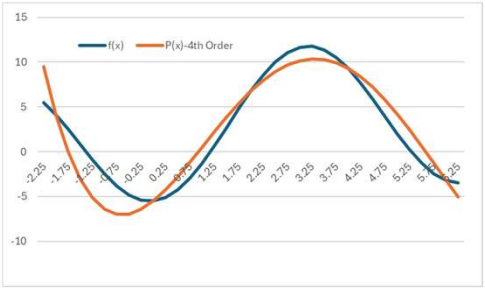
$$L_0(x) = \frac{1}{384}x^4 - \frac{1}{32}x^3 + \frac{11}{96}x^2 - \frac{1}{8}x$$

$$L_1(x) = -\frac{1}{96}x^4 + \frac{5}{48}x^3 - \frac{5}{24}x^2 - \frac{5}{12}x + 1$$

$$L_2(x) = \frac{1}{64}x^4 - \frac{1}{8}x^3 + \frac{1}{16}x^2 + \frac{3}{4}x$$

$$L_3(x) = -\frac{1}{96}x^4 + \frac{1}{16}x^3 + \frac{1}{24}x^2 - \frac{1}{4}x$$

$$L_4(x) = \frac{1}{384}x^4 - \frac{1}{96}x^3 - \frac{1}{96}x^2 + \frac{1}{24}x$$



$f(x_0) = 4.14$
$f(x_1) = -5.5$
$f(x_2) = 6.74$
$f(x_3) = 9.26$
$f(x_4) = -3.18$

The provided data points correspond to the following function.
$\boldsymbol{f(x) = 2\ln(x+3.5) - 8\cos(x)}$

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) + L_4(x)f(x_4)$$

$$= 0.06865x^4 - 0.93292x^3 + 2.46042x^2 + 4.38167x - 5.5$$

Results:
P(-1)=−6.42
P(1)=0.478
P(5)=4.21

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Example (Assignment 9) Find f(-1), f(1) and f(5) given that f(x) passes through (-2, 4.14), (0, -5.5), (2, 6.74), (4, 9.26) and (6, -3.18).

Solution 3: Use two quadratic polynomials: Lagrange interpolating polynomials to assign two 2nd order polynomials to the given points

$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_{21}(x)f(x_2)$ using points (-2, 4.14), (0, -5.5), (2, 6.74)

$P_2(x) = L_{22}(x)f(x_2) + L_3(x)f(x_3) + L_4(x)f(x_4)$ using points (2, 6.74), (4, 9.26) (6, -3.18).

$$L_0(x) = \frac{(x-0)}{(-2-0)}\frac{(x-2)}{(-2-2)} = \frac{1}{8}(x^2 - 2x)$$
$$L_1(x) = \frac{(x+2)}{(0+2)}\frac{(x-2)}{(0-2)} = \frac{1}{4}(4 - x^2)$$
$$L_{21}(x) = \frac{(x+2)}{(2+2)}\frac{(x-0)}{(2-0)} = \frac{1}{8}(x^2 + 2x)$$
$$L_{22}(x) = \frac{(x-4)}{(2-4)}\frac{(x-6)}{(2-6)} = \frac{1}{8}(x^2 - 10x + 24)$$
$$L_3(x) = \frac{(x-2)}{(4-2)}\frac{(x-6)}{(4-6)} = -\frac{1}{4}(x^2 - 8x + 12)$$
$$L_4(x) = \frac{(x-2)}{(6-2)}\frac{(x-4)}{(6-4)} = \frac{1}{8}(x^2 - 6x + 8)$$

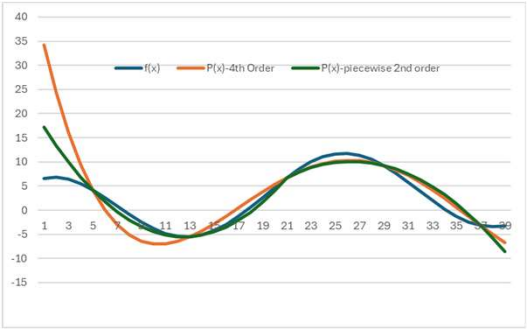
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$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_{21}(x)f(x_2)$$
$$= \frac{4.14}{8}(x^2 - 2x) - \frac{5.5}{4}(4 - x^2) + \frac{6.74}{8}(x^2 + 2x)$$
$$= 2.735x^2 + 0.65x - 5.5$$

$\boldsymbol{P_1(-1) = -3.415}$ and $\boldsymbol{P_1(1) = -2.115}$

$$P_2(x) = L_{22}(x)f(x_2) + L_3(x)f(x_3) + L_4(x)f(x_4)$$
$$= \frac{6.74}{8}(x^2 - 10x + 24) - \frac{9.26}{4}(x^2 - 8x + 12) - \frac{3.18}{8}(x^2 - 6x + 8)$$
$$= -1.87x^2 + 12.48x - 10.74$$

$\boldsymbol{P_2(5) = 4.91}$



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