# MT308 Numerical Analysis

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## Iterative Methods to Solve Linear Systems

- Iterative methods are useful if the size of matrix is huge and if there are many 0 elements. During the iterations matrix is not changed so sparcity is preserved
- Starts with an initial approximation for the solution vector ($x^0$)
- At each iteration updates the x vector by using the sytem Ax=b
- Each iteration involves a matrix-vector product
- Write the system Ax=b in an equivalent form **x=Ex+f**
- (like x=g(x) for fixed-point iteration)
- Starting with $x^0$, generate a sequence of approximations $\{x^k\}$ iteratively by **$x^{k+1}=Ex^k+f$**

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# Jacobi's Method

The Jacobi iterative method is obtained by solving the ith equation in Ax = b for $x_i$ to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left( -\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n.$$

For each k ≥ 1, generate the components $x_i^{(k)}$ of $x^{(k)}$ from the components of $x^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left( -a_{ij}x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n.$$

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Apply the Jacobi Method to the system $3u + v = 5, u + 2v = 5$.

$$u = \frac{5 - v}{3}$$
$$v = \frac{5 - u}{2}.$$

use the initial guess $(u_0, v_0) = (0, 0)$

The two equations are iterated:

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}.$$

The Jacobi method converges toward the solution [1,2].

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Apply the Jacobi Method to the system $u + 2v = 5, 3u + v = 5$.

$$u = 5 - 2v$$
$$v = 5 - 3u.$$

use the initial guess $(u_0, v_0) = (0, 0)$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5 - 2v_0 \\ 5 - 3u_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 - 2v_1 \\ 5 - 3u_1 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$
$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 - 2(-10) \\ 5 - 3(-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}$$

Diverges

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The Jacobi iteration method for solving a linear system converges if any of the following conditions is satisfied:

1. $A$ **is strictly diagonally dominant**
   For every row $i$:

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

   This is a *sufficient* condition, not necessary.

2. $A$ **is symmetric positive definite (SPD)**
   If $A$ is symmetric ($A = A^T$) and $x^T Ax > 0$ for all $x \neq 0$, Jacobi converges.
   (However, Gauss-Seidel generally performs better on SPD systems.)

A square matrix is called **positive definite** if it is **symmetric and all its** eigenvalues λ are positive, that is λ >0.

3. **The spectral radius** $\rho(G_J) < 1$
   Where $G_J = D^{-1}(L + U)$ is the Jacobi iteration matrix.
   This is the **necessary and sufficient condition for convergence.**
   $\rho(G_J)$ is the largest absolute value of the eigenvalues of $G_J$.

**Spectral radius** of a square matrix $A$:
$\rho(G_j) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of} G_j\}$

$$x_{k+1} = -D^{-1}(L + U)x_k + D^{-1}b.$$

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Remember from MT104 Linear Algebra:

Definition:

Let $A$ be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a **nonzero vector** for which

$$AX = \lambda X$$

for some scalar $\lambda$. Then $\lambda$ is called an **eigenvalue** of the matrix $A$ and $X$ is called an **eigenvector** of $A$ associated with $\lambda$, or a $\lambda$-eigenvector of $A$.

Theorem:

Let $A$ be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$.
Then $\lambda$ is an eigenvalue of $A$ and thus there exists a nonzero vector $X \in \mathbb{C}^n$ such that $AX = \lambda X$.

$$AX - \lambda X = 0$$
or
$$(A - \lambda I)X = 0$$

Matlab Function eig():

- e = eig(A) returns a column vector containing the eigenvalues of the square matrix A.
- [V,D] = eig(A) returns a diagonal matrix D of eigenvalues and matrix V whose columns are the corresponding right eigenvectors, so that A*V = V*D.

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Consider the example we solved: $\begin{matrix} u + 2v = 5 \\ 3u + v = 5 \end{matrix}$ *or* $\begin{matrix} 3u + v = 5 \\ u + 2v = 5 \end{matrix}$

This problem can be written as

Strictly diagonally dominant

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Both of the equations are exactly the same.

But when an iterative solution is obtained one of them does not give a convergent solution.

We need to use the one that is strictly diagonally dominant to obtain a convergent solution with the Jacobi iteration

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```
function x = jacobi_it(A,b,x0,tol)
n = length(b);
x = x0;
while norm(A*x - b, inf) > tol
    x_new = zeros(n,1);
    for i = 1:n
        s = A(i,1:i-1)*x(1:i-1) + A(i,i+1:n)*x(i+1:n);
        x_new(i) = (b(i) - s)/A(i,i);
    end
    x = x_new;
end
end
```

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# Gauss Seidel Method

It is similar to the Jacobi Method
In the Gauss–Seidel Method, updated values of the unknowns are used at each step, even if the updating occurs in the current step.

Remember the example

$$3u + v = 5, u + 2v = 5$$

$$u = \frac{5 - v}{3}$$
$$v = \frac{5 - u}{2}.$$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$
$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix}$$
$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}.$$

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Example:

Apply the Gauss–Seidel Method to the system

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

The Gauss–Seidel iteration is

$$u_{k+1} = \frac{4 - v_k + w_k}{3}$$
$$v_{k+1} = \frac{1 - 2u_{k+1} - w_k}{4}$$
$$w_{k+1} = \frac{1 + u_{k+1} - 2v_{k+1}}{5}.$$

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# Solution:

Starting with $x_0 = [u_0, v_0, w_0] = [0, 0, 0]$, we calculate

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{4-0-0}{3} = \frac{4}{3} \\ \frac{1-8/3-0}{4} = -\frac{5}{12} \\ \frac{1+4/3+5/6}{5} = \frac{19}{30} \end{bmatrix} \approx \begin{bmatrix} 1.3333 \\ -0.4167 \\ 0.6333 \end{bmatrix}$$

and

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{101}{60} \\ -\frac{3}{4} \\ \frac{251}{300} \end{bmatrix} \approx \begin{bmatrix} 1.6833 \\ -0.7500 \\ 0.8367 \end{bmatrix}.$$

The system is strictly diagonally dominant, and therefore the iteration will converge to the solution $[2, -1, 1]$. ◀

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