MT308 Numerical Analysis

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Solving Nonlinear Equations f(x)=0

- **Bracketing Methods**

If the function is continuous on the bracket

Bisection Method
False-Position Method

And if the bracket includes only one root, it always converges

- **Open Methods**

➢Do NOT require an interval
➢Start from one or two initial guesses
➢Usually faster than bracketing methods
➢Do NOT guarantee convergence

Fixed Point Iteration
Newton-Raphson
Secant Method

Open methods are very sensitive to the initial value(s). A poor initial guess may lead to divergence, convergence to the wrong root, or oscillation.

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Open Methods

- Open methods use one or two **initial guesses**, not brackets.
- Often **converge faster** than ***bracketing methods***.
- But there is a **risk of divergence** if improperly applied.
- We'll learn 3 well known methods:
  - **Fixed Point Iteration**
  - **Newton-Raphson**
  - **Secant Method**

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Fixed point

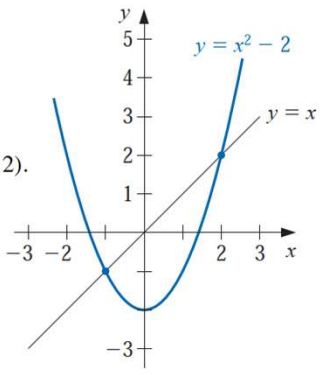
- A fixed point for a function is a number at which the value of the function does not change when the function is applied.
- The number p is a fixed point for a given function g if g(p) = p.

Example:

Determine any fixed points of the function $g(x) = x^2 - 2$.

Solution A fixed point p for g has the property that

$$p = g(p) = p^2 - 2 \quad \text{which implies that} \quad 0 = p^2 - p - 2 = (p + 1)(p - 2).$$



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Fixed Point Iteration

We can use this method if the problem, f(x)=0, can be reformulated as x=g(x). Then an iteration is defined by the following expression:

$$x_{n+1}=g(x_n)$$

for n=0,1,2,... where x_0 is the initial guess.

The process continues until one of the following convergence criteria is satisfied

$$|x_n - x_{n-1}| \leq \varepsilon \;\; or \; f(x_n) \leq \delta$$

This iteration converges if the following conditions are true:

1. g(x) is continuous
2. $\left|\frac{dg(x)}{dx}\right| < 1$

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Example: Solve $x^2 - x - 30 = 0$ (Find the root of $f(x) = x^3 - x - 30$)

Solution: At the first glance we may re-formulate the problem for fixed point iteration as follows $x = g(x) = x^2 - 30$

$x_{n+1} = g(x_n) = {x_n}^2 - 30$				
iteration	x_0=-20	x_0=-5.5	x_0=5.5	x_0=15
0	-20	-5.5	5.5	15
1	370	0.25	0.25	195
2	136870	-29.937		
3	18733396870	866.253		
4	3.5094E+20	750365		
5	1.23159E+41	5.63E+1		
6	1.51681E+82	3.17E+2		
7	2.3007E+164	1.01E+4		
8	#NUM!	1.01E+94	1.01E+94	1.3E+293

None of the iterations presented here converges.
The absolute value of the derivative of g(x) is greater than 1 around the roots:

$|g(x)| = 2x$>1 at x=-5 and x=6

!! Therefore, We need alternative formulations for this problem

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Example: Solve $x^2 - x - 30 = 0$ (Find the root of $f(x) = x^2 - x - 30$)

Solution: We may re-formulate the problem by $x^2 = x + 30$, and then $x = \pm\sqrt{x + 30}$ (There are two roots one positive and one negative)

$x_{n+1} = -\sqrt{x_n + 30}$			$x_{n+1} = \sqrt{x_n + 30}$		
-20	-5.5	20	-20	-5.5	20
-3.16228	-4.94975	-7.07106781	3.162278	4.949747	7.071068
-5.18051	-5.00502	-4.78841646	5.75867	5.911831	6.088601
-4.98192	-4.9995	-5.02111377	5.979855	5.992648	6.007379
-5.00181	-5.00005	-4.99788818	5.998321	5.999387	6.000615
-4.99982	-4.99999	-5.00021118	5.99986	5.999949	6.000051
-5.00002	-5	-4.99997888	5.999988	5.999996	6.000004
-5	-5	-5.00000211	5.999999	6	6
-5	-5	-4.99999979	6	6	6
-5	-5	-5.00000002	6	6	6
-5	-5	-5	6	6	6

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Example: Solve $f(x) = e^x - 1 - 2x = 0$ with an accuracy of $\varepsilon = 10^{-2}$

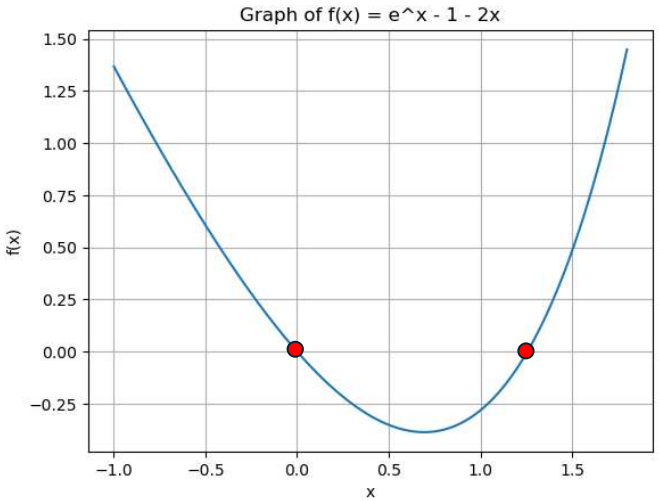
Solution: It is important to identify where the root is located.

From $\frac{df(x)}{dx} = e^x - 2 = 0$, it can be found that x=ln(2) is a minimum point for f(x).

f(ln(2))<0 and f(0)=0

There is one more root in the range 1<x<2.

We may try $x = g(x) = \frac{e^x - 1}{2}$



But $\frac{dg(x)}{dx} = \frac{e^x}{2}$ and it does not satisfy the condition $|g'(x)| < 1$ *if* 1<x<2

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Example: Solve $f(x) = e^x - 1 - 2x = 0$ with an accuracy of $\varepsilon = 10^{-2}$

Solution: Since $x_{n+1} = g(x_n) = \frac{e^{x_n}-1}{2}$ does not converge, we need to find alternative fixed-point expressions. For example,

$$f(x) = e^x - 1 - 2x = 0 \Rightarrow e^x = 1 + 2x$$

$$\Rightarrow x = \ln(1 + 2x) = g(x)$$

Where $g'(x) = \frac{2}{(1+2x)}$ and $|g'(x)| < 1$ if 1<x<2 .

Therefore the following iteration converges:

$$x_{n+1} = g(x_n) = \ln(1 + 2x_n)$$

n	xn	error
0	1.5	|xn+1-xn|
1	1.3863	Error
2	1.3278	0.0585
3	1.2962	0.0315
4	1.2788	0.0174
5	1.2691	0.0097
6	1.2636	0.0055
7	1.2605	0.0031
8	1.2588	0.0018
9	1.2578	0.0010
10	1.2572	0.0006
11	1.2569	0.0003
12	1.2567	0.0002
13	1.2566	1.05E-04
14	1.2565	5.98E-05
15	1.2565	3.40E-05

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Newton-Raphson's Method

- Newton's (or the Newton-Raphson) method is one of the most powerful and well-known numerical methods for solving a root-finding problem.
- The Newton-Raphson method is a powerful iterative numerical technique for finding the roots of a differentiable function accurately and quickly. It starts with an initial guess x_0 and improves it using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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Suppose that f(x) ∈ C^2[a, b]. Let p_0 ∈ [a, b] be an approximation to p such that f '(p_0) ≠ 0 and | p − p0| is "small". Consider the first Taylor polynomial for f (x) expanded about p0 and evaluated at x = p.

$$f(p) = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

where ξ(p) lies between p and p0. If f (p) = 0, then,

$$0 = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

and considering that | p − p0| is very small,

$$0 \approx f(p_0) + (p - p_0) f'(p_0). \quad \Rightarrow \quad p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

By rearranging the last expression,

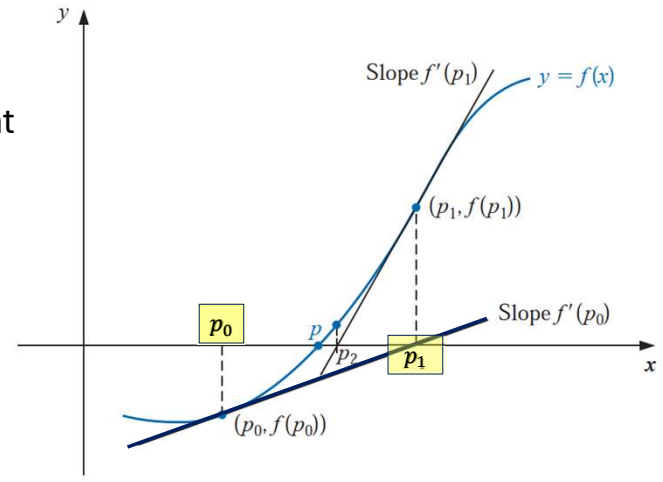
$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

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Graphical Explanation of Newton Raphson's method

- Start with an initial guess (p_0)
- Sketch a tangent line at p_0.
- Find the point where the tangent line intersects the x-axis (p_1).
- Use p_1 to find p_2.
- Repeat the same process by using p_n to find p_{n+1}.

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$



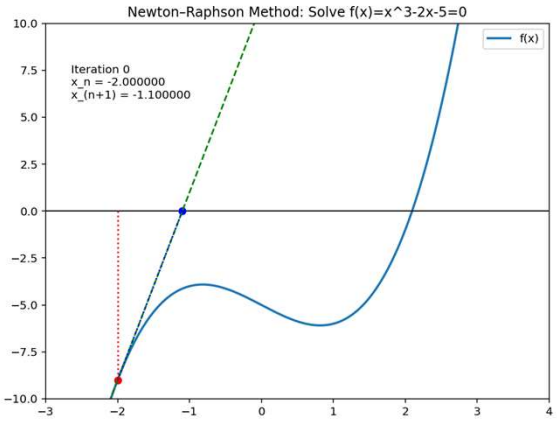
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Example: Find the root of $f(x) = x^3 - 2x - 5$

Since $f'(x) = 3x^2 - 2$ then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

n	x_n	$f(x_n)$	$f'(x_n)$	Error
0	-2	-9	10	$\lvert x_{n+1} - x_n \rvert$
1	-1.1	-4.131	1.63	0.9
2	1.4344	-4.918	4.172	2.534
3	2.6131	7.616	18.484	1.179
4	2.2010	1.261	12.534	0.412
5	2.1004	0.066	11.235	0.101
6	2.0946	2.2E-04	11.162	5.86E-03
7	2.0946	2.4E-09	11.161	1.94E-05



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Example: Use Newton Raphson method to solve $f(x) = e^x - 1 - 2x = 0$ with an accuracy of $\varepsilon = 10^{-2}$

Solution: Since $f'(x) = e^x - 2$

The Newton-Raphson iteration formula is,

$$x_{n+1} = x_n - \frac{\exp(x_n) - 1 - 2x_n}{\exp(x_n) - 2}$$

n	xn	f(xn)	f'(xn)	Error
0	1.5	0.482	2.481689	|xn-xn+1|
1	1.305903	0.079	1.69102	0.194097
2	1.259059	0.004	1.522105	0.047
3	1.256439	1.21E-05	1.51289	2.6E-03
4	1.256431	1.12E-10	1.512862	8.0E-06
5	1.256431	0	1.512862	7.4E-11

Remember the fixed-point iteration:
$x_{n+1} = g(x_n) = \ln(1 + 2x_n)$

n	Newton-Rapson xn	Newton-Rapson f(xn)	Fixed Point xn	Fixed Point f(xn)
0	1.5	0.482	1.5	0.482
1	1.305903	0.079	1.386294	0.227
2	1.259059	0.004	1.327761	0.117
3	1.256439	1.21E-05	1.296239	0.063
4	1.256431	1.12E-10	1.278842	0.035
5	1.256431	0	1.26911	0.019
6	1.256431	0	1.263624	0.011

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Example: Find the root of $f(x) = cos(x) - x = 0$, using (a) The fixed-point iteration, and (b) Newton-Raphson Method

a) Fixed-Point Iteration

$x_{n+1} = g(x_n) = \cos(x_n)$

n	xn	|f(xn)|	n	xn	|f(xn)|
0	0.5	0.377583	13	0.74033	0.002083
1	0.877583	0.23857	14	0.738246	0.001404
2	0.639012	0.163673	15	0.73965	0.000945
3	0.802685	0.107907	16	0.738705	0.000637
4	0.694778	0.073418	17	0.739341	0.000429
5	0.768196	0.04903	18	0.738912	0.000289
6	0.719165	0.03319	19	0.739201	0.000195
7	0.752356	0.022275	20	0.739007	0.000131
8	0.730081	0.015039	21	0.739138	8.83E-05
9	0.74512	0.010114	22	0.73905	5.95E-05
10	0.735006	0.00682	23	0.739109	4.01E-05
11	0.741827	0.004591	24	0.739069	2.7E-05
12	0.737236	0.003094	25	0.739096	1.82E-05

b) Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = =x_n + \frac{cos(x_n) - x_n}{sin(x_n) + 1}$$

n	xn	f(xn)	f'(xn)	|f(xn)|
0	0.5	0.377583	-1.47943	0.377583
1	0.755222	-0.0271	-1.68545	0.027103
2	0.739142	-9.5E-05	-1.67365	9.46E-05
3	0.739085	-1.2E-09	-1.67361	1.18E-09
4	0.739085	0	-1.67361	0
5	0.739085	0	-1.67361	0

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The Secant Method

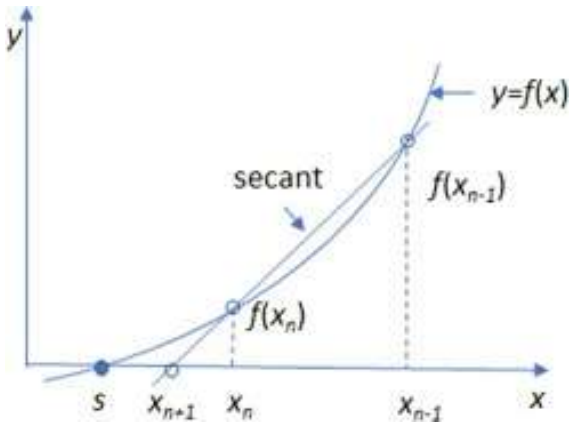
The word secant is derived from the Latin word secan, which means to cut. The secant method uses a secant line, a line joining two points that cut the curve, to approximate a root.

The Secant method is an alternative to the Newton-Raphson method when the derivative of f(x) is unknown or difficult to calculate.

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

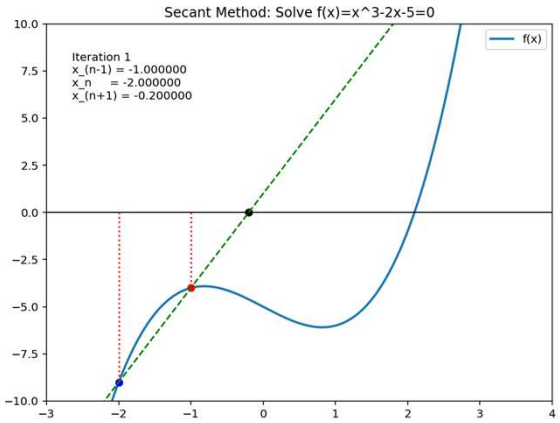
$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$



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Example: Use the secant method to find the root of $f(x) = x^3 - 2x - 5$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$



n	xn	f(xn)	Error
-1	-1	-4	|xn-xn-1|
0	-2	-9	1
1	-0.2	-4.61	1.8
2	1.688525	-3.56	1.888525
3	8.126552	515.43	6.438027
4	1.732721	-3.26	6.39383
5	1.772947	-2.97	0.040225
6	2.18484	1.06	0.411894
7	2.076601	-0.20	0.108239
8	2.093665	-0.01	0.017064
9	2.094561	0.00	0.000895
10	2.094551	-5E-08	9.03E-06
11	2.094551	-2.5E-13	4.5E-09
12	2.094551	0	2.26E-14

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Assignment 3 (Section 1):
Write a Matlab/Octave script to find the root of $f(x) = x^2 - x - 30$ by using the fixed point iteration

- Upload the *.m file you wrote
- Run the script and upload the image of the Command Window
- Exercise Problem: Find the root of the following function:

$$f(x) = x^4 - x^2 - \frac{1}{x} + \ln(x)$$

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Assignment 3 (Section 2):
Write a MATLAB/Octave function to find the root of

$$f(x) = x^4 - x^2 - \frac{1}{x} + \ln(x)$$

using Newton-Raphson Method

- Upload the *.m file you wrote
- Run this function for two different initial guesses and upload the image of the Command Window

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