G.A.U. Faculty Of Engineering

MT308 Numerical Analysis

Duration: 90 Minutes Midterm (optional) Exam 30th of July. 2025

Note: Use at least 2 decimal places.

1. Given that $f(x) = x^3 + x - 20$, (a) Develop an iteration formula and show that this iteration is convergent. (b) Calculate the first three iterations with any appropriate guess. (c) What is the absolute error done at (b)?

   Solution: (A) The fixed-point iteration: $x_{n+1} = g(x_n) = \sqrt[3]{20 - x_n}$.

   Since $\left|\frac{dg(x)}{dx}\right| = \left|\frac{1}{3}(20-x)^{-2/3}\right| < 1 \rightarrow \frac{1}{3} < (20-x)^{2/3} \rightarrow \frac{1}{27} < (20-x)^2$ then this iteration is convergent if the root is in the range

   $x < \left(20 - \frac{1}{\sqrt{27}}\right)$ x or $\left(20 - \frac{1}{\sqrt{27}}\right) < x$

   Let $x_0 = 3$ and the absolute error at the third iteration is $|x_3 - x_2| = 1.063 \times 10^{-3}$

n	x_n	$g(x_n)$
0	3	2.571282
1	2.571282	2.592717
2	2.592717	2.591654
3	2.591654	2.591707
4	2.591707	2.591704

   (B) Newton Raphson iteration:

   $x_{n+1} = x_n - \frac{x_n^3 + x_n - 20}{3x_n^2 + 1}$

   Using $x_0 = 3$

   the absolute error at the third iteration is $|x_3 - x_2| = 9.39 \times 10^{-4}$

n	x_n	$f(x_n)$	$f'(x_n)$
0	3	10	28
1	2.642857	1.102405	21.95408
2	2.592643	0.019865	21.16539
3	2.591704	6.85E-06	21.1508
4	2.591704	8.21E-13	21.15079

2. (a) Use the bisection method to find a root of $f(x) = \ln(2x) + x^3$ within the interval 0.2<x<0.6. Continue for 4 iterations (b) Calculate the absolute error at the 4th iterations.

   Solution:

n	a_n	$x_n = \frac{b_n - a_n}{2}$	b_n	$f(a_n)$	$f(x_n)$	$f(b_n)$
0	0.2	0.4	0.6	-0.908	-0.159	0.398
1	0.4	0.5	0.6	-0.159	0.125	0.398
2	0.4	0.45	0.5	-0.159	-0.014	0.125
3	0.45	0.475	0.5	-0.014	0.056	0.125
4	0.45	0.4625	0.475	-0.014	0.021	0.056

Absolute error= 0.0125

3. Use Gaussian elimination method to solve the following system:
   4x+y+2z=5
   -x-y+4z=-17
   -4x+8y-z=8

Solution:

$$\begin{bmatrix} 4 & 1 & 2 & 5 \\ -1 & -1 & 4 & -17 \\ -4 & 8 & -1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 1 & 2 & 5 \\ 0 & -0.75 & 4.5 & -15.75 \\ 0 & 9 & 1 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 1 & 2 & 5 \\ 0 & -0.75 & 4.5 & -15.75 \\ 0 & 0 & 55 & -176 \end{bmatrix}$$

By back substitution z=-176/55=-3.2, y=-(-15.75-4.5*(-3.2))/0.75=1.8, and x=(5-1.8-2*(-3.2))/4 =2.4.

4. Use Jacobi iteration to solve the following system (Continue for 3 iterations, use initial guess (1, 1,-2))
   4x+y+2z=5
   -x-y+4z=-17
   -4x+8y-z=8
   Solution: In order to obtain a convergent iteration we need to reorganize the order of the equations: (The coefficients on the diagonal must be greater than the sum of the absolute values of the other coefficients in the same row.)
   4x+y+2z=5 then $x_{n+1} = (5 - y_n - 2z_n)/4$
   -4x+8y-z=8 then $y_{n+1} = (8 + 4x_n + z_n)/8$
   -x-y+4z=-17 then $z_{n+1} = (-17 + x_n + y_n)/4$

n	x_n	y_n	z_n
0	1	1	-2
1	2	1.25	-3.75
2	2.8125	1.5313	-3.4375
3	2.5859	1.9766	-3.1641

5. Use the Taylor Series to calculate $f(x) = e^{(2-x)}$ around $x_0 = 2$. What is the relative error if only 5 non-zero terms of the polynomial is used to approximate f(6).
   Solution: $f(x) = e^{(2-x)}$, $f'(x) = -e^{(2-x)}$, $f''(x) = e^{(2-x)}$, $f'''(x) = -e^{(2-x)}$
   $f^{(n)}(x) = (-1)^n e^{(2-x)}$ and $f^{(n)}(x_0) = f^{(n)}(2) = (-1)^n$ then

$$f(x) = e^{(2-x)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(x-2)^n$$
$$= 1 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^4$$
$$- \frac{1}{120}(x-2)^5 \ldots$$

$f(6) \cong 1 - 4 + \frac{1}{2}4^2 - \frac{1}{6}(4)^3 + \frac{1}{24}4^4 = 5$

Note: This is not an accurate answer since only 5 nonzero terms are used.

$f(6) = e^{(2-6)} = e^{-4} = 0.018316$

6. Write a matlab script to solve Question 4.

Solution:

```
x0=1;
y0=1;
z0=-2;
for n=1:5,
  x=(5-y0-2*z0)/4;
  y=(8+4*x0+z0)/8;
  z=(x0-17+y0)/4;
  x0=x;
  y0=y;
  z0=z;
  printf("%d %f %f %f\n",n,x,y,z);
end
```

Output:

```
1 2.000000 1.250000 -3.750000
2 2.812500 1.531250 -3.437500
3 2.585938 1.976562 -3.164062
4 2.337891 1.897461 -3.109375
5 2.330322 1.780273 -3.191162
```

7. Write a matlab function named taylor1 which accepts two arguments, x and N, where N is the number of nonzero terms of the Taylor Polynomial to approximate $f(x) = e^{(2-x)}$. The function should return the f(x) value.

Solution: There is factorial() function in matlab as can be seen in the first program. The second function is written assuming the student does not know the factorial function.

```
function f=taylor1(x,N)
  f=1;
  for n=1:N
    f=f+(-1)^n*(x-2)^n/factorial(n);
  end
```

Output:

```
>> taylor1(6,4)
ans = 5
>> taylor1(6,10)
ans = 0.096720
>> taylor1(6,40)
ans = 0.018316
```

```
function f=taylor1(x,N)
  f=1;
  f1=1;
  for n=1:N
    f1=f1*n;
    f=f+(-1)^n*(x-2)^n/f1;
  end
```

8. Write a matlab function to solve question 1 using the secant method.

```
function x=secant1(x0,x1,err)
  f=@(x) x^3+x-20;
  while abs(f(x1)-f(x0))>err
    x=x1-f(x1)*(x1-x0)/(f(x1)-f(x0));
    x0=x1;
    x1=x;
  end
end
```

Output:

```
>> secant1(2,3,1E-6)
ans = 2.5917
```

- Bisection Method

$$x_{\text{mid}} = \frac{x_{\text{lower}} + x_{\text{upper}}}{2}$$

- Newton-Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Secant Method

$$x_{i+1} = x_i - f(x_i)\left(\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right)$$

- Gauss-Seidel Iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

- Jacobi Iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right)$$

- Taylor Polynomial Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$